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FROM RANDOM PROCESSES TO GENERALIZED FIELDS: A UNIFIED APPROACH TO STOCHASTIC INTEGRATION

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ABSTRACT. The paper studies stochastic integration with respect to Gaussian processes and fields. It is more convenient to work with a field than a process: by definition, a field is a collection of stochastic integrals for a class of deterministic integrands. The problem is then to extend the definition to random integrands. An orthogonal decomposition of chaos space of the random field leads to two such extensions, corresponding to the Itô-Skorokhod and the Stratonovich integrals, and provides an efficient tool to study these integrals, both analytically and numerically. For a Gaussian process, a natural definition of the integral follows from a canonical correspondence between random processes and a special class of random fields.

1. INTRODUCTION

While stochastic integral with respect to a standard Brownian motion is a well-studied object, integration with respect to other Gaussian processes is currently an area of active research, and the fractional Brownian motion is receiving most of the attention [1, 3, 5, 6, 7, 11, 12, 16, etc.] The objective of this paper is to define and investigate stochastic integrals with respect to arbitrary Gaussian processes and fields using chaos expansion. The motivation comes from the paper by Alòs et al. [2] and the book by P. Major [14].

In [2], the authors study stochastic integration with respect to the Gaussian process $\int_0^t K(t, s) dW(s)$, where K is a suitable kernel function and W is a standard Brownian motion. In [14], the author studies stochastic integration with respect to generalized Gaussian fields. While [2] and [14] pursue different goals and work with different objects, generalized fields and the chaos expansion, appearing in both [2] and [14], are the unifying ideas.

A generalized Gaussian field \mathfrak{X} over a Hilbert space \mathbf{H} is a continuous linear mapping $f \mapsto \mathfrak{X}(f)$ from \mathbf{H} to the space of Gaussian random variables. The corresponding chaos space $\mathbb{H}_{\mathfrak{X}}$ is the Hilbert space of square integrable random variables that are measurable with respect to the sigma-algebra generated by $\mathfrak{X}(f)$, $f \in \mathbf{H}$. The chaos expansion is an orthogonal decomposition of $\mathbb{H}_{\mathfrak{X}}$: given an orthonormal basis

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$\{\xi_m, m \geq 1\}$ in $\mathbb{H}_{\mathfrak{X}}$, a square integrable \mathbf{H} -valued random variable η has a chaos expansion $\eta = \sum_{m \geq 1} \eta_m \xi_m$, with $\eta_m = \mathbb{E}(\eta \xi_m) \in \mathbf{H}$.

The definition of a generalized Gaussian field \mathfrak{X} already provides the stochastic integral $\mathfrak{X}(f)$ for non-random $f \in \mathbf{H}$. As a result, given the chaos expansion of a random element η from $\mathbb{H}_{\mathfrak{X}}$, the definition of the stochastic integral $\mathfrak{X}(\eta)$ requires an extension of the linearity property of \mathfrak{X} to linear combinations with random coefficients. Two “natural” extensions of this property lead to the Itô-Skorokhod and the Stratonovich stochastic integrals; see Definition 4.6 below. Both integrals can be expressed using the Malliavin derivative and divergence operator on the chaos space $\mathbb{H}_{\mathfrak{X}}$.

Even for non-random f , when there is no difference between the Itô-Skorokhod and the Stratonovich interpretations of $\mathfrak{X}(f)$, there are often several ways of computing $\mathfrak{X}(f)$. It is most convenient to work with a white noise over \mathbf{H} , that is, a zero-mean generalize Gaussian field such that $\mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g)) = (f, g)_{\mathbf{H}}$ for all $f, g \in \mathbf{H}$. It turns out that, for every zero-mean Gaussian field \mathfrak{X} over \mathbf{H} , there exists a different (usually larger) Hilbert space \mathbf{H}' such that \mathfrak{X} is a white noise over \mathbf{H}' . Moreover, the space \mathbf{H}' is uniquely determined by \mathfrak{X} . On the other hand, every zero-mean Gaussian field \mathfrak{X} over \mathbf{H} can be written in the form $\mathfrak{X}(f) = \mathfrak{B}(\mathcal{K}^* f)$, $f \in \mathbf{H}$, where \mathcal{K}^* is a bounded linear operator on \mathbf{H} and \mathfrak{B} is a white noise over \mathbf{H} , although this white noise representation of \mathfrak{X} is not necessarily unique. Thus, different white noise representations of \mathfrak{X} lead to different formulas for computing $\mathfrak{X}(f)$, and the chaos expansion is an efficient way for deriving those formulas. In particular, for both deterministic and random f , chaos expansion provides an explicit formula for $\mathfrak{X}(f)$ in terms of the Fourier coefficients of the integrand f .

To define stochastic integral with respect to a Gaussian process $X = X(t)$, $t \in [0, T]$, we construct a Hilbert space \mathbf{H}_X and a white noise \mathfrak{B} over \mathbf{H}_X such that $X(t) = \mathfrak{B}(\chi_t)$, where χ_t is the characteristic function of the interval $[0, t]$. The space \mathbf{H}_X is uniquely determined by X ; for example, the Wiener process on $(0, T)$ has $\mathbf{H}_X = L_2((0, T))$. Then the equality

$$(1.1) \quad \int_0^T f(s) dX(s) = \mathfrak{B}(f), \quad f \in \mathbb{H}_{\mathfrak{B}},$$

is a canonical definition of the stochastic integral with respect to X .

In some situations, given a Gaussian process $X = X(t)$, $t \in [0, T]$, it is possible to find a generalized Gaussian field \mathfrak{X} over a Hilbert space \mathbf{H} so that $X(t) = \mathfrak{X}(\chi_t)$. Even though \mathfrak{X} is not necessarily a white noise over \mathbf{H} , the resulting definition of the stochastic integral,

$$\int_0^T f(t) dX(t) = \mathfrak{X}(f),$$

coincides with the (1.1), while the space \mathbf{H} can be more convenient for computations than the space \mathbf{H}_X . For example, fractional Brownian motion with the Hurst parameter bigger than $1/2$ has a rather complicated space \mathbf{H}_X , but can be represented using a generalized Gaussian field over $\mathbf{H} = L_2((0, T))$.

The paper is organized as follows. Section 2 provides the definition and properties of generalized Gaussian fields and establishes connections with the Gaussian processes.

Section 3 introduces the chaos expansion and the Wick product, both necessary for the definition and analysis in Section 4 of the stochastic integrals with random integrands.

The main contributions of the paper are:

- (1) Two white noise representations of a zero-mean generalized Gaussian field (Theorem 2.4);
- (2) A connection between generalized Gaussian fields over $L_2((0, T))$ and processes that are representable in the form $\int_0^t K(t, s)dW(s)$ (Theorem 2.7);
- (3) Chaos expansions of the Itô-Skorokhod and Stratonovich integrals (Theorem 4.7);
- (4) Investigation of the equation $u(t) = 1 + \int_0^t u(s)dX(s)$ for a class of Gaussian random processes X (Theorem 4.8).

In particular, we establish the following result.

Theorem 1.1. *Let \mathfrak{X} be a zero-mean generalized Gaussian field over $L_2((0, T))$ and $X(t) = \mathfrak{X}(\chi_t)$. Then the solution of the Itô equation*

$$u(t) = 1 + \int_0^t u(s)dX(s)$$

is unique in the class of square integrable \mathcal{F}^X -measurable processes and is given by

$$u(t) = e^{X(t) - \frac{1}{2}\mathbb{E}X^2(t)}.$$

2. GENERALIZED GAUSSIAN FIELDS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and \mathbf{V} , a linear topological space over the real numbers \mathbb{R} . Everywhere in this paper, we assume that the probability space is rich enough to support all the random elements we might need.

Definition 2.1. (a) *A generalized random field over \mathbf{V} is a mapping $\mathfrak{X} : \Omega \times \mathbf{V} \rightarrow \mathbb{R}$ with the following properties:*

- (1) *For every $f \in \mathbf{V}$, $\mathfrak{X}(f) = \mathfrak{X}(\cdot, f)$ is a random variable;*
- (2) *For every $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathbf{V}$, $\mathfrak{X}(\alpha f + \beta g) = \alpha \mathfrak{X}(f) + \beta \mathfrak{X}(g)$;*
- (3) *If $\lim_{n \rightarrow \infty} f_n = f$ in the topology of \mathbf{V} , then $\lim_{n \rightarrow \infty} \mathfrak{X}(f_n) = \mathfrak{X}(f)$ in probability.*

(b) *A generalized random field \mathfrak{X} is called*

- **zero-mean**, if $\mathbb{E}\mathfrak{X}(f) = 0$ for all $f \in \mathbf{V}$;
- **Gaussian**, if the random variable $\mathfrak{X}(f)$ is Gaussian for every $f \in \mathbf{V}$.

FOR EXAMPLE, if $W = W(t)$, $0 \leq t \leq T$, is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathfrak{X}(f) = \int_0^T f(t)dW(t)$ is a zero-mean generalized Gaussian field over $L_2((0, T))$; note that

$$(2.1) \quad \mathbb{E}|\mathfrak{X}(f_n) - \mathfrak{X}(f)|^2 = \int_0^T |f_n(t) - f(t)|^2 dt.$$

More generally, if \mathcal{M} is a bounded linear operator on $L_2((0, T))$, then

$$(2.2) \quad \mathfrak{X}(f) = \int_0^T (\mathcal{M}f)(t) dW(t)$$

is a zero-mean generalized Gaussian field over $L_2((0, T))$. In fact, by Theorem 2.6(b) below, every zero-mean generalized Gaussian field over $L_2((0, T))$ can be represented in the form (2.2) with suitable \mathcal{M} and W . We will also see that the fractional Brownian motion on $[0, T]$ with Hurst parameter bigger than $1/2$ can be interpreted as a zero-mean generalized Gaussian field over $L_2((0, T))$.

Let \mathbf{H} be a real Hilbert space with inner product $(\cdot, \cdot)_{\mathbf{H}}$ and norm $\|\cdot\|_{\mathbf{H}} = \sqrt{(\cdot, \cdot)_{\mathbf{H}}}$. The following is a useful property of generalized Gaussian fields over \mathbf{H} .

Theorem 2.2. *For every zero-mean generalized Gaussian field \mathfrak{X} over a Hilbert space \mathbf{H} , there exists a unique bounded linear self-adjoint operator \mathcal{R} on \mathbf{H} such that*

$$(2.3) \quad \mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g)) = (\mathcal{R}f, g)_{\mathbf{H}}, \quad f, g \in \mathbf{H}.$$

Proof. Equality (2.3) implies that the operator \mathcal{R} , if exists, must be unique. To establish existence of \mathcal{R} , denote by $\tilde{\mathbf{H}}$ the Hilbert space $L_2(\Omega, \mathcal{F}, \mathbb{P})$ of square integrable random variables. By Definition 2.1, the mapping $f \mapsto \mathfrak{X}(f)$ defines a continuous linear operator from \mathbf{H} to $\tilde{\mathbf{H}}$ (recall that, for Gaussian random variables, convergence in probability implies mean-square convergence). Therefore, there exists a positive number C such that, for every $f \in \mathbf{H}$,

$$(2.4) \quad \|\mathfrak{X}(f)\|_{\tilde{\mathbf{H}}}^2 = \mathbb{E}|\mathfrak{X}(f)|^2 \leq C\|f\|_{\mathbf{H}}^2.$$

Fix $f \in \mathbf{H}$ and consider the linear functional F on \mathbf{H} defined by $F(g) = \mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g))$. By (2.4), this functional is bounded:

$$|F(g)| = |\mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g))| \leq \sqrt{\mathbb{E}|\mathfrak{X}(f)|^2} \sqrt{\mathbb{E}|\mathfrak{X}(g)|^2} \leq C\|f\|_{\mathbf{H}} \|g\|_{\mathbf{H}},$$

and therefore, by the Riesz Representation Theorem, there exists a unique $h_f \in \mathbf{H}$ such that $F(g) = \mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g)) = (h_f, g)_{\mathbf{H}}$. Define the operator \mathcal{R} by $\mathcal{R}f = h_f$. By construction, this operator is linear; it is bounded by (2.4). A bounded linear operator satisfying (2.3) automatically satisfies $(\mathcal{R}f, g)_{\mathbf{H}} = (f, \mathcal{R}g)_{\mathbf{H}}$ and is therefore self-adjoint. \square

Definition 2.3. (a) The operator \mathcal{R} from Theorem 2.2 is called the **covariance operator** of \mathfrak{X} . (b) A **white noise** over \mathbf{H} is a zero-mean generalized Gaussian field with the covariance operator equal to the identity operator.

Note that if \mathcal{R} is the covariance operator of \mathfrak{X} and $\mathcal{R}f = 0$, then $\mathfrak{X}(f) = 0$ (\mathbb{P} -a.s.). Writing $\ker(\mathcal{R})$ to denote the zero-space of \mathcal{R} , we have a direct sum decomposition $\mathbf{H} = \ker(\mathcal{R}) \oplus \ker(\mathcal{R})^{\perp}$, where $\ker(\mathcal{R})^{\perp}$ is the orthogonal complement of $\ker(\mathcal{R})$. As a result, if $f = f_1 + f_2$, with $f_1 \in \ker(\mathcal{R})$, $f_2 \in \ker(\mathcal{R})^{\perp}$, then $\mathfrak{X}(f) = \mathfrak{X}(f_2)$. We say that the Gaussian field is **non-degenerate** if $\ker(\mathcal{R}) = 0$.

We show next that every zero-mean Gaussian random field over a Hilbert space can be reduced to a white noise in two different ways.

Theorem 2.4. (a) For every zero-mean generalized Gaussian field \mathfrak{X} over a Hilbert space \mathbf{H} , there exist a bounded linear operator \mathcal{K} on \mathbf{H} and a white noise \mathfrak{B} over \mathbf{H} so that $\mathcal{K}\mathcal{K}^*$ is the covariance operator of \mathfrak{X} and, for every $f \in \mathbf{H}$,

$$(2.5) \quad \mathfrak{X}(f) = \mathfrak{B}(\mathcal{K}^*f);$$

as usual, \mathcal{K}^* denotes the adjoint of \mathcal{K} .

(b) For every zero-mean non-degenerate generalized Gaussian field \mathfrak{X} over a Hilbert space \mathbf{H} , there exists a Hilbert space $\mathbf{H}_{\mathcal{R}}$ such that \mathbf{H} is continuously embedded into $\mathbf{H}_{\mathcal{R}}$ and \mathfrak{X} extends to a white noise over $\mathbf{H}_{\mathcal{R}}$.

Proof. (a) By construction, the covariance operator \mathcal{R} of a generalized Gaussian field is non-negative definite, bounded, and self-adjoint on \mathbf{H} . Indeed, \mathcal{R} is bounded on \mathbf{H} by Theorem 2.2, and, for every f and g from \mathbf{H} , we have

$$\begin{aligned} (\mathcal{R}f, f)_{\mathbf{H}} &= \mathbb{E}(\mathfrak{X}(f))^2 \geq 0; \\ (\mathcal{R}f, g)_{\mathbf{H}} &= \mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g)) = \mathbb{E}(\mathfrak{X}(g)\mathfrak{X}(f)) = (\mathcal{R}g, f)_{\mathbf{H}} = (f, \mathcal{R}g)_{\mathbf{H}}. \end{aligned}$$

Therefore, by a standard result from functional analysis (see, for example, [8, page 923]) there exists a bounded linear operator \mathcal{K} on \mathbf{H} such that $\mathcal{R} = \mathcal{K}\mathcal{K}^*$; this operator \mathcal{K} is not necessarily unique.

Next, let $\mathcal{K}^*(\mathbf{H}) = \{\mathcal{K}^*f, f \in \mathbf{H}\}$ be the range of \mathcal{K}^* , which is a closed linear subspace of \mathbf{H} . Denote by $\mathcal{K}^*(\mathbf{H})^{\perp}$ the orthogonal complement of $\mathcal{K}^*(\mathbf{H})$ in \mathbf{H} . Then, for every $f \in \mathbf{H}$, there exists a unique pair (f_1, f_2) , with $f_1 \in \mathbf{H}$, $f_2 \in \mathcal{K}^*(\mathbf{H})^{\perp}$, such that $f = \mathcal{K}^*f_1 + f_2$. This orthogonal decomposition of f implies

$$(2.6) \quad \|f\|_{\mathbf{H}}^2 = \|\mathcal{K}^*f_1\|_{\mathbf{H}}^2 + \|f_2\|_{\mathbf{H}}^2,$$

and

$$(2.7) \quad \mathbb{E}(\mathfrak{X}(f_1))^2 = (\mathcal{K}\mathcal{K}^*f_1, f_1)_{\mathbf{H}} = \|\mathcal{K}^*f_1\|_{\mathbf{H}}^2.$$

Define

$$\tilde{\mathfrak{X}}(f) = \mathfrak{X}(f_1).$$

Then $\tilde{\mathfrak{X}}$ is a generalized Gaussian field over \mathbf{H} : if $\lim_{n \rightarrow \infty} \|f_n - f\|_{\mathbf{H}}^2 = 0$, then, by (2.6) and (2.7),

$$\lim_{n \rightarrow \infty} \mathbb{E}(\tilde{\mathfrak{X}}(f) - \tilde{\mathfrak{X}}(f_n))^2 = \lim_{n \rightarrow \infty} \mathbb{E}(\mathfrak{X}(f_1 - f_{1,n}))^2 = \lim_{n \rightarrow \infty} \|\mathcal{K}^*(f_1 - f_{1,n})\|_{\mathbf{H}}^2 = 0.$$

Let $\overline{\mathfrak{B}}$ be a white noise over \mathbf{H} , independent of \mathfrak{X} . The same arguments show that $\tilde{\mathfrak{B}}$, defined by

$$\tilde{\mathfrak{B}}(f) = \overline{\mathfrak{B}}(f_2),$$

is a generalized Gaussian field over \mathbf{H} . Define

$$\mathfrak{B}(f) = \tilde{\mathfrak{X}}(f) + \tilde{\mathfrak{B}}(f).$$

Then \mathfrak{B} is a generalized Gaussian field over \mathbf{H} , being a sum of two independent generalized Gaussian fields over \mathbf{H} , and, by definition, $\mathfrak{B}(\mathcal{K}^*f) = \tilde{\mathfrak{X}}(\mathcal{K}^*f) = \mathfrak{X}(f)$.

Moreover, if $f = \mathcal{K}^* f_1 + f_2$, $g = \mathcal{K}^* g_1 + g_2$, then

$$\begin{aligned} \mathbb{E}(\mathfrak{B}(f)\mathfrak{B}(g)) &= \mathbb{E}(\mathfrak{X}(f_1)\mathfrak{X}(g_1)) + \mathbb{E}(\overline{\mathfrak{B}}(f_2)\overline{\mathfrak{B}}(g_2)) \\ &= (\mathcal{K}\mathcal{K}^* f_1, g_1)_{\mathbf{H}} + (f_2, g_2)_{\mathbf{H}} = (\mathcal{K}^* f_1, \mathcal{K}^* g_1)_{\mathbf{H}} + (f_2, g_2)_{\mathbf{H}} = (f, g)_{\mathbf{H}}, \end{aligned}$$

where the first equality follows from the independence of \mathfrak{X} and $\widetilde{\mathfrak{B}}$, and the last, from $(\mathcal{K}^* f_1, g_2)_{\mathbf{H}} = (\mathcal{K}^* g_1, f_2)_{\mathbf{H}} = 0$. Thus, \mathfrak{B} is a white noise over \mathbf{H} , and the proof of (2.5) is complete.

(b) Define $\mathbf{H}_{\mathcal{R}}$ as the closure of \mathbf{H} with respect to the inner product $(f, g)_{\mathbf{H}_{\mathcal{R}}} = (\mathcal{R}f, g)_{\mathbf{H}}$. For $f \in \mathbf{H}$, $\|f\|_{\mathbf{H}_{\mathcal{R}}}^2 = (\mathcal{R}f, f)_{\mathbf{H}} \leq C\|f\|_{\mathbf{H}}^2$, which implies a dense continuous embedding of \mathbf{H} into $\mathbf{H}_{\mathcal{R}}$. By definition, for $f, g \in \mathbf{H}$, $\mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g)) = (\mathcal{R}f, g)_{\mathbf{H}} = (f, g)_{\mathbf{H}_{\mathcal{R}}}$. As a result, if $f \in \mathbf{H}_{\mathcal{R}}$ and $\lim_{n \rightarrow \infty} \|f_n - f\|_{\mathbf{H}_{\mathcal{R}}}^2 = 0$, with $f_n \in \mathbf{H}$, then

$$\lim_{m, n \rightarrow \infty} \mathbb{E}(\mathfrak{X}(f_m) - \mathfrak{X}(f_n))^2 = \lim_{m, n \rightarrow \infty} \|f_m - f_n\|_{\mathbf{H}_{\mathcal{R}}}^2 = 0,$$

so that $\lim_{n \rightarrow \infty} \mathfrak{X}(f_n)$ exists in the mean-square and is therefore a Gaussian random variable. We then define $\mathfrak{X}(f) = \lim_{n \rightarrow \infty} \mathfrak{X}(f_n)$. The value of $\mathfrak{X}(f)$ does not depend on the sequence $\{f_n, n \geq 1\}$ approximating f , because

$$\lim_{n \rightarrow \infty} \mathbb{E}(\mathfrak{X}(f) - \mathfrak{X}(f_n))^2 = \|f - f_n\|_{\mathbf{H}_{\mathcal{R}}}^2 = 0.$$

Also,

$$\mathbb{E}(\mathfrak{X}(f)\mathfrak{X}(g)) = \lim_{n \rightarrow \infty} \mathbb{E}(\mathfrak{X}(f_n)\mathfrak{X}(g_n)) = \lim_{n \rightarrow \infty} (f_n, g_n)_{\mathbf{H}_{\mathcal{R}}} = (f, g)_{\mathbf{H}_{\mathcal{R}}},$$

meaning that this extension of \mathfrak{X} is a white noise over $\mathbf{H}_{\mathcal{R}}$. \square

Remark 2.5. (a) If \mathfrak{X} is non-degenerate and $\mathcal{R} : \mathbf{H} \rightarrow \mathbf{H}$ is onto, then \mathcal{R} has a bounded inverse and $\mathbf{H}_{\mathcal{R}} = \mathbf{H}$. (b) If $\ker \mathcal{R}$ is non-trivial, then we can define $\mathbf{H}_{\mathcal{R}}$ as the closure of the factor space $\mathbf{H}/\ker(\mathcal{R})$ with respect to the inner product $(\bar{f}, \bar{g})_{\mathbf{H}_{\mathcal{R}}} = (\mathcal{R}f, g)_{\mathbf{H}}$, where \bar{f} is the equivalence class of f in $\mathbf{H}/\ker(\mathcal{R})$. Direct computations show that the generalized random field \mathfrak{B} over $\mathbf{H}/\ker(\mathcal{R})$, defined by

$$\mathfrak{B}(\bar{f}) = \mathfrak{X}(f), \quad f \in \mathbf{H},$$

extends to a white noise over $\mathbf{H}_{\mathcal{R}}$.

We will now discuss several connections between generalized Gaussian fields and Gaussian processes. In what follows, I denotes either an interval $[0, T]$, or the half-line $[0, +\infty)$, or all of \mathbb{R} .

Denote by $\chi_t = \chi_t(s)$ the characteristic function of the interval $[0, t]$:

$$(2.8) \quad \chi_t(s) = \begin{cases} 1, & 0 \leq s \leq t; \\ 0, & \text{otherwise.} \end{cases}$$

With this definition, $\chi_{t_2}(s) - \chi_{t_1}(s)$ is the characteristic function of the interval $(t_1, t_2]$, $t_2 > t_1$.

Theorem 2.6. (a) If \mathfrak{B} is a white noise over $L_2(I)$, then $B(t) = \mathfrak{B}(\chi_t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and, for every $f \in L_2(I)$, we have

$$(2.9) \quad \mathfrak{B}(f) = \int_I f(s) dB(s).$$

(b) For every zero-mean non-degenerate generalized Gaussian field \mathfrak{X} over $L_2(I)$, there exist a bounded linear operator \mathcal{K}^* on $L_2(I)$ and a standard Brownian motion $W = W(t)$ such that, for every $f \in L_2(I)$,

$$(2.10) \quad \mathfrak{X}(f) = \int_I (\mathcal{K}^* f)(s) dW(s).$$

Proof. (a) Direct computations show that $B = B(t)$ has all the properties of the standard Brownian motion. In particular,

$$\mathbb{E}(B(t_1)B(t_2)) = \int_0^T \chi_{t_1}(s)\chi_{t_2}(s)ds = \min(t_1, t_2).$$

Next, if $s_0 < s_1 < \dots < s_N$ is a finite collection of points in I and $f(s) = \sum_{k=1}^N a_k(\chi_{s_k}(s) - \chi_{s_{k-1}}(s))$ is a (non-random) step function, then the linearity property of the generalized random field \mathfrak{B} implies

$$\mathfrak{B}(f) = \sum_{k=1}^N a_k(\mathfrak{B}(\chi_{s_k}) - \mathfrak{B}(\chi_{s_{k-1}})) = \int_I f(s) dB(s).$$

For general f , the result then follows after passing to the limit, using the continuity property of the generalized random field \mathfrak{B} and the L_2 -isometry of the stochastic integral.

(b) This follows from part (a) and from Theorem 2.4. \square

Given a zero-mean generalized Gaussian field \mathfrak{X} over $L_2(I)$, we define its **associated process** $X(t)$, $t \in I$, by

$$(2.11) \quad X(t) = \mathfrak{X}(\chi_t).$$

Clearly, $X(t)$ is a Gaussian process. Let \mathcal{K}^* be the operator from Theorem 2.6 and define the kernel function $K_{\mathfrak{X}} = K_{\mathfrak{X}}(t, s)$ by

$$(2.12) \quad K_{\mathfrak{X}}(t, s) = (\mathcal{K}^* \chi_t)(s).$$

It then follows from (2.10) that

$$(2.13) \quad X(t) = \int_I K_{\mathfrak{X}}(t, s) dW(s)$$

for some standard Brownian motion W . Let us emphasize that, while every kernel $K(t, s)$ with minimal integrability properties can define a Gaussian process according to (2.13), only a process associated with a generalized field over $L_2(I)$ has a kernel defined according to (2.12), where \mathcal{K}^* is a bounded operator on $L_2(I)$. Recall that the definition of a generalized field (Definition 2.1) includes a certain continuity property, and this property translates into addition structure of the kernel function in the representation of the associated process.

Now assume that we are given a Gaussian process $X(t)$ defined by (2.13) with some kernel $K_{\mathfrak{X}}(t, s)$. *We are not assuming that $K_{\mathfrak{X}}$ has the form (2.12).* In what follows, we discuss sufficient conditions on $K_{\mathfrak{X}}(t, s)$ ensuring that $X(t)$ is the associated process of a generalized Gaussian field \mathfrak{X} over $L_2(I)$, that is, representation (2.12) does indeed hold with some bounded linear operator \mathcal{K}^* on $L_2(I)$. For that, we need to recover the operator \mathcal{K}^* from the kernel $K_{\mathfrak{X}}(t, s)$. By linearity, if (2.12) holds and if $s_0 < s_1 < \dots < s_N$ are points in I and

$$(2.14) \quad f(s) = \sum_{k=0}^{N-1} a_k (\chi_{s_{k+1}}(s) - \chi_{s_k}(s))$$

is a step function, then

$$(2.15) \quad \mathcal{K}^* f(s) = \sum_{k=0}^{N-1} a_k (K_{\mathfrak{X}}(s_{k+1}, s) - K_{\mathfrak{X}}(s_k, s)).$$

To extend (2.15) to continuous functions f , the kernel $K_{\mathfrak{X}}(t, s)$ must have bounded variation as a function of t ; if this is indeed the case, then (2.15) implies that, for every smooth compactly supported function f on I ,

$$(2.16) \quad \mathcal{K}^* f(s) = \int_I f(t) K_{\mathfrak{X}}(dt, s).$$

The assumption about the bounded variation of the kernel is used extensively in [2], and the connection with generalized fields shows that this assumption is very natural. It now follows that if the partial derivative $\partial K_{\mathfrak{X}}(t, s)/\partial t$ exists and is square integrable over $I \times I$, then \mathcal{K}^* , as defined by (2.16), extends to a bounded linear operator on $L_2(I)$.

Let us now assume that $I = [0, T]$ and the process $X(t)$ defined by (2.13) is **non-anticipating**, i.e. adapted to the filtration $\{\mathcal{F}_t^W, 0 \leq t \leq T\}$ generated by the Brownian motion $W(s)$. Then $K_{\mathfrak{X}}(t, s) = 0$ for $s > t$ and (2.13) becomes

$$(2.17) \quad X(t) = \int_0^t K_{\mathfrak{X}}(t, s) dW(s).$$

Note that in this case we have

$$(2.18) \quad \mathbb{E}(X(t)X(s)) = \int_0^{\min(t,s)} K_{\mathfrak{X}}(t, \tau) K_{\mathfrak{X}}(s, \tau) d\tau.$$

This is the type of processes studied in [2], and for such processes, formula (2.15) and the conditions for the continuity of the corresponding operator \mathcal{K}^* must be modified as follows.

Theorem 2.7. *Assume that $I = [0, T]$ and the process $X(t)$ defined by (2.13) is non-anticipating.*

(a) If f is a step function (2.14), then

$$(2.19) \quad \begin{aligned} \mathcal{K}^* f(s) &= \sum_{i=0}^{N-1} (\chi_{s_{i+1}}(s) - \chi_{s_i}(s)) \left(a_i K_{\mathfrak{X}}(s_{i+1}, s) \right. \\ &\quad \left. + \sum_{k=i+1}^{N-1} a_k (K_{\mathfrak{X}}(s_{k+1}, s) - K_{\mathfrak{X}}(s_k, s)) \right). \end{aligned}$$

(b) If the function $K_{\mathfrak{X}}(\cdot, s)$ has bounded variation for every s and $\lim_{\delta \rightarrow 0, \delta > 0} K_{\mathfrak{X}}(s+\delta, s) = K_{\mathfrak{X}}(s^+, s)$ exists for all $s \in (0, T)$, then

$$(2.20) \quad \mathcal{K}^* f(s) = K_{\mathfrak{X}}(s^+, s) f(s) + \int_s^T f(t) K_{\mathfrak{X}}(dt, s)$$

for every continuous on $[0, T]$ function f .

(c) If the function $K_{\mathfrak{X}}(t, s)$ has the following properties

- (1) K is continuous and non-negative for $0 \leq s \leq t \leq T$, and $\sup_{0 < t < T} K(t, t) \leq K_0$;
- (2) $K^{(1)}(t, s) = \partial K(t, s) / \partial t$ is non-negative for $0 < s < t < T$ and there exists a number $K_1 = K_1(T)$ such that

$$(2.21) \quad \sup_{0 < t < T} \int_0^t K(T, s) K^{(1)}(t, s) ds \leq K_1(T),$$

then the corresponding operator \mathcal{K}^* defined by equation (2.20) is bounded on $L_2((0, T))$ and the operator norm $\|\mathcal{K}^*\|$ of \mathcal{K}^* satisfies

$$(2.22) \quad \|\mathcal{K}^*\|^2 \leq \begin{cases} 2(K_0^2 + K_1), & \text{if } K_0 > 0; \\ K_1, & \text{if } K_0 = 0. \end{cases}$$

Proof. (a) By assumption, $K_{\mathfrak{X}}(t, s) = 0$ for $s > t$. Fix an s such that $s \in (s_j, s_{j+1}]$ for some $j = 0, \dots, N-1$. By (2.15) we have for this value of s

$$\begin{aligned} \mathcal{K}^* f(s) &= \sum_{k=0}^{N-1} a_k (K_{\mathfrak{X}}(s_{k+1}, s) - K_{\mathfrak{X}}(s_k, s)) \\ &= a_j K_{\mathfrak{X}}(s_{j+1}, s) + \sum_{k=j+1}^{N-1} a_k (K_{\mathfrak{X}}(s_{k+1}, s) - K_{\mathfrak{X}}(s_k, s)). \end{aligned}$$

Since $\chi_{s_{k+1}}(s) - \chi_{s_k}(s)$ is the characteristic function of the interval $(s_k, s_{k+1}]$, (2.19) follows.

(b) Under the additional assumptions on the kernel $K_{\mathfrak{X}}$, (2.20) follows from (2.19) after passing to the limit $\max_{j=0, \dots, N-1} |s_{j+1} - s_j| \rightarrow 0$.

(c) Let g be a smooth compactly supported function on $(0, T)$. It follows from (2.20) that

$$\mathcal{K}^* g(s) = K(s, s) g(s) ds + \int_s^T \frac{\partial K(\tau, s)}{\partial \tau} g(\tau) d\tau = K(s, s) g(s) ds + \int_s^T K^{(1)}(\tau, s) g(\tau) d\tau.$$

To estimate the L_2 -norm of the integral, we use the Cauchy-Schwartz inequality and the properties of $K^{(1)}$:

$$\begin{aligned}
& \int_0^T \left| \int_s^T K^{(1)}(\tau, s) g(\tau) d\tau \right|^2 ds = \int_0^T \left| \int_s^T [K^{(1)}(\tau, s)]^{1/2} [K^{(1)}(\tau, s)]^{1/2} g(\tau) d\tau \right|^2 ds \\
& \leq \int_0^T \int_s^T K^{(1)}(\tau, s) d\tau \int_s^T K^{(1)}(\tau, s) g^2(\tau) d\tau ds \\
& \leq \int_0^T (K(T, s) - K(s, s)) \int_s^T K^{(1)}(\tau, s) g^2(\tau) d\tau ds \\
& \leq \int_0^T \left(\int_0^\tau K(T, s) K^{(1)}(\tau, s) ds \right) g^2(\tau) d\tau \leq K_1(T) \|g\|_{L_2((0, T))}^2.
\end{aligned}$$

□

We remark that in [2] relation (2.19) is used to *define* the operator \mathcal{K}^* corresponding to a non-anticipating process $X(t)$. Using the connection with the generalized fields, Theorem 2.7 shows that this definition is reasonable.

The main example covered by part (c) of Theorem 2.7 is the fractional Brownian motion W^H on $[0, T]$ with the Hurst parameter $H > 1/2$. Indeed, it is known (see [15, Section 5.1.3]) that in this case W^H has representation (2.17) with

$$K_{\mathfrak{X}}(t, s) = C_H \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t (\tau - s)^{H-\frac{3}{2}} \tau^{H-\frac{1}{2}} d\tau,$$

where

$$C_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{\frac{1}{2}}$$

and Γ is the Gamma-function. Clearly, $K_{\mathfrak{X}}(s, s) = 0$ and so $K_0 = 0$. Then somewhat lengthy computations show that

$$(2.23) \quad K_1(T) = \frac{H(2H-1)\Gamma(H-\frac{1}{2})}{\Gamma(H+\frac{1}{2})} T^{2H-1}.$$

The bound $K_1(T)$ is asymptotically optimal: since $\lim_{x \rightarrow 0^+} x\Gamma(x) = \lim_{x \rightarrow 0^+} \Gamma(1+x) = 1$, the right-hand side of (2.23) converges to 1 as $H \searrow \frac{1}{2}$, and if $H = 1/2$, then W^H is the standard Brownian motion and \mathcal{K}^* is the identity operator, which corresponds to $\|\mathcal{K}^*\| = 1$.

The following theorem establishes a connection between a zero-mean Gaussian process and white noise.

Theorem 2.8. *For every zero-mean Gaussian process $X = X(t)$, $t \in I \subseteq \mathbb{R}$ with covariance function $R(t, s) = \mathbb{E}(X(t)X(s))$, there exist*

- (1) *a Hilbert space \mathbf{H}_R containing the indicator functions χ_t ;*
- (2) *a white noise \mathfrak{B} over \mathbf{H}_R*

such that $X(t) = \mathfrak{B}(\chi_t)$.

Proof. Let the Hilbert space \mathbf{H}_R be the closure of the set of the step functions with respect to the inner product

$$(\chi_{t_1}, \chi_{t_2})_{\mathbf{H}_R} = R(t_1, t_2).$$

Define a generalized Gaussian field \mathfrak{B} over \mathbf{H}_R by setting

$$(2.24) \quad \mathfrak{B}(\chi_t) = X(t),$$

and then extending by linearity and continuity to all of \mathbf{H}_R . With this definition, \mathfrak{B} is a white noise over \mathbf{H}_R . \square

By analogy with (2.9), if \mathfrak{X} is a generalized Gaussian field over a Hilbert space \mathbf{H} of functions or generalized functions on I , and $X(t)$ is the associated process of \mathfrak{X} , then $\int_I f(s) dX(s)$ can be an alternative notation for $\mathfrak{X}(f)$.

The space \mathbf{H}_R from Theorem 2.8 appears in [2] and is different from reproducing kernel Hilbert space used in [16, Section 6]. If $X(t)$ is the associated process of a zero-mean non-degenerate generalized Gaussian field \mathfrak{X} over $\mathbf{H} = L_2(I)$, and \mathcal{R} is the covariance operator of \mathfrak{X} , then $R(t, s) = (\mathcal{R}\chi_t, \chi_s)_{L_2(I)}$ and the space \mathbf{H}_R coincides with $\mathbf{H}_{\mathcal{R}}$ from Theorem 2.4.

Given a covariance function R , an explicit characterization of the space \mathbf{H}_R is impossible without additional assumptions about R . For example, in [2], representation

$$R(t, s) = \int_0^{\min(t, s)} K(t, \tau) K(s, \tau) d\tau,$$

is used, along with various assumptions about the kernel K . If $I = [0, T]$ and $R(t, s) = \min(t, s)$, then $(\chi_{t_1}, \chi_{t_2})_{\mathbf{H}_R} = (\chi_{t_1}, \chi_{t_2})_{L_2((0, T))}$. That is, for the Wiener process, $\mathbf{H}_R = L_2((0, T))$.

Let us summarize the main results of this section:

- Every zero-mean generalized Gaussian random field over \mathbf{H} with covariance operator \mathcal{R} has two white noise representations: over the Hilbert space $\mathbf{H}_{\mathcal{R}}$ and over the original space \mathbf{H} ;
- Every zero-mean Gaussian random process with covariance function R is the associated process of a white noise over the Hilbert space \mathbf{H}_R .

3. CHAOS DECOMPOSITION AND THE WICK PRODUCT

Let \mathfrak{X} be a zero-mean generalized Gaussian field over a real Hilbert space \mathbf{H} , on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From now on, we assume that the space \mathbf{H} is separable. Denote by $\mathcal{F}^{\mathfrak{X}}$ the sigma-algebra generated by the random variables $\mathfrak{X}(f)$, $f \in \mathbf{H}$.

Definition 3.1. (a) *The chaos space generated by \mathfrak{X} is the collection of all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ that are square integrable and $\mathcal{F}^{\mathfrak{X}}$ -measurable. This chaos space will be denoted by $\mathbb{H}_{\mathfrak{X}}$.*

(b) *The first chaos space generated by \mathfrak{X} is the sub-space of $\mathbb{H}_{\mathfrak{X}}$, consisting of the random variables $\mathfrak{X}(f)$, $f \in \mathbf{H}$. The first chaos space will be denoted by $\mathbb{H}_{\mathfrak{X}}^{(1)}$.*

It follows that $\mathbb{H}_{\mathfrak{X}}$ is a Hilbert space with inner product $(\xi, \eta)_{\mathbb{H}_{\mathfrak{X}}} = \mathbb{E}(\xi\eta)$, and $\mathbb{H}_{\mathfrak{X}}^{(1)}$ is a Hilbert sub-space of $\mathbb{H}_{\mathfrak{X}}$. Moreover, the space $\mathbb{H}_{\mathfrak{X}}^{(1)}$ is separable: if $\{\bar{f}_1, \bar{f}_2, \dots\}$ is a dense countable set in \mathbf{H} , then the collection of all finite linear combinations of $\mathfrak{X}(\bar{f}_i)$ with rational coefficients is a dense countable set in $\mathbb{H}_{\mathfrak{X}}^{(1)}$.

Our next objective is to show how an orthonormal basis in $\mathbb{H}_{\mathfrak{X}}^{(1)}$ leads to an orthonormal basis in $\mathbb{H}_{\mathfrak{X}}$. We will need some additional constructions.

For an integer $n \geq 0$, the n -th Hermite polynomial $H_n = H_n(t)$ is defined by

$$(3.1) \quad H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}.$$

In particular, $H_0(t) = 1$, $H_1(t) = t$, $H_2(t) = t^2 - 1$, $H_3(t) = t^3 - 3t$, etc. Note that $H_n(t) = t^n + \dots$, that is, H_n is a polynomial of degree n and the leading coefficient is always equal to one. It is well known that if ξ is a standard Gaussian random variable, then

$$(3.2) \quad \mathbb{E}(H_n(\xi)H_m(\xi)) = \begin{cases} n!, & n = m; \\ 0, & n \neq m. \end{cases}$$

In fact, the collection $\{H_n(\xi), n \geq 0\}$ is an orthonormal basis in the space of square integrable, \mathcal{F}^ξ -measurable random variables.

Next, denote by \mathcal{I} the collection of multi-indices, that is, sequences $\alpha = \{\alpha_k, k \geq 1\} = \{\alpha_1, \alpha_2, \dots\}$ with the following properties:

- each α_k is a non-negative integer: $\alpha_k \in \{0, 1, 2, \dots\}$.
- only finitely many of α_k are non-zero: $|\alpha| := \sum_{k=1}^{\infty} \alpha_k < \infty$.

The set \mathcal{I} is countable, being a countable union of countable sets. By ϵ_n we denote the multi-index $\alpha = \{\alpha_k, k \geq 1\}$ with $\alpha_k = 1$ if $n = k$ and $\alpha_k = 0$ otherwise. For $\alpha \in \mathcal{I}$, we will use the notation

$$\alpha! := \alpha_1! \alpha_2! \dots$$

Let $\{\xi_1, \xi_2, \dots\}$ be an ordered countable collection of random variables. For $\alpha \in \mathcal{I}$ define random variables ξ_α as follows:

$$(3.3) \quad \xi_\alpha = \prod_{k \geq 1} \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}},$$

where H_{α_k} is α_k -th Hermite polynomial (3.1). For example, $\alpha = (0, 2, 0, 1, 3, 0, 0, \dots)$ has three non-zero entries $\alpha_2 = 2$, $\alpha_4 = 1$, and $\alpha_5 = 3$, so that

$$\xi_\alpha = \frac{H_2(\xi_2)}{\sqrt{2!}} \cdot H_1(\xi_4) \cdot \frac{H_3(\xi_5)}{\sqrt{3!}} = \frac{\xi_2^2 - 1}{\sqrt{2}} \xi_4 \frac{\xi_5^3 - 3\xi_5}{\sqrt{6}}.$$

The product on the right hand side of (3.3) is finite for every $\alpha \in \mathcal{I}$. Note also that $\xi_k = H_1(\xi_k) = \xi_{\epsilon_k}$ and, more generally, $H_n(\xi_k) = \sqrt{n!} \xi_{n\epsilon_k}$.

The following theorem has been known for some time in various forms. In the particular case when $\mathfrak{X}(f) = \int_0^T f(t)dW(t)$, this theorem is the main result of the paper [4] by Cameron and Martin; see also [9, Theorem 1.9] and [10, Theorem 2.2.3]. The formulation and proof below are similar to [14, Theorem 2.1].

Theorem 3.2. *Let $\{\xi_1, \xi_2, \dots\}$ be an orthonormal basis in $\mathbb{H}_{\mathfrak{X}}^{(1)}$. Then the collection $\Xi = \{\xi_\alpha, \alpha \in \mathcal{I}\}$ is an orthonormal basis in $\mathbb{H}_{\mathfrak{X}}$: for every $\eta \in \mathbb{H}_{\mathfrak{X}}$ we have*

$$\eta = \sum_{\alpha \in \mathcal{I}} \left(\mathbb{E}(\eta \xi_\alpha) \right) \xi_\alpha, \quad \mathbb{E}\eta^2 = \sum_{\alpha \in \mathcal{I}} \left(\mathbb{E}(\eta \xi_\alpha) \right)^2.$$

Proof. Recall that \mathfrak{X} is a Gaussian random field. As a result, an orthonormal basis in $\mathbb{H}_{\mathfrak{X}}^{(1)}$ is a collection of standard Gaussian random variables ξ_k , $k \geq 1$, that are uncorrelated, hence independent. Then property (3.2) of Hermite polynomials implies that Ξ is an orthonormal system.

Next, denote by H_{ξ_k} the Hilbert space of square integrable random variables that are measurable with respect to the sigma-algebra generated by ξ_k . Consider the product space $H_\infty = \prod_{k=1}^{\infty} H_{\xi_k}$. By the definition of the product topology, it follows that the collection $\{\xi_\alpha, \alpha \in \mathcal{I}\}$ is an orthonormal basis in this product space. We also note that, by construction, the sigma-algebra $\mathcal{F}^{\mathfrak{X}}$ is generated by the random variables ξ_1, ξ_2, \dots , and therefore, for every $\mathcal{F}^{\mathfrak{X}}$ -measurable random variable η , there exists a measurable, real-valued function F on the measurable space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ with the property $\eta = F(\xi_1, \xi_2, \dots)$. This establishes a one-to-one correspondence between the product space H_∞ and the chaos space $\mathbb{H}_{\mathfrak{X}}$, and completes the proof. \square

By definition, the space $\mathbb{H}_{\mathfrak{X}}^{(1)}$ is generated by ξ_α with $|\alpha| = 1$. More generally, we define $\mathbb{H}_{\mathfrak{X}}^{(N)}$, the *N-th chaos space* of \mathfrak{X} , as the closure in $\mathbb{H}_{\mathfrak{X}}$ of the linear span of ξ_α with $|\alpha| = N$: $\eta \in \mathbb{H}_{\mathfrak{X}}^{(N)}$ if and only if $\eta = \sum_{\alpha \in \mathcal{I}, |\alpha|=N} c_\alpha \xi_\alpha$ for some real numbers c_α satisfying $\sum_\alpha |c_\alpha|^2 < \infty$. By Theorem 3.2 we have the *chaos decomposition* of $\mathbb{H}_{\mathfrak{X}}$:

$$(3.4) \quad \mathbb{H}_{\mathfrak{X}} = \bigoplus_{N=0}^{\infty} \mathbb{H}_{\mathfrak{X}}^{(N)} = \mathbb{H}_{\mathfrak{X}}^{(0)} \oplus \mathbb{H}_{\mathfrak{X}}^{(1)} \oplus \mathbb{H}_{\mathfrak{X}}^{(2)} \oplus \dots$$

Proposition 3.3. *For each N , the space $\mathbb{H}_{\mathfrak{X}}^{(N)}$ does not depend on the choice of the basis in $\mathbb{H}_{\mathfrak{X}}^{(1)}$.*

Proof. Since the polynomials H_0, H_1, \dots, H_N are orthogonal with respect to the Gaussian measure on \mathbb{R} , these polynomials are linearly independent. Therefore, for each $N \geq 0$, the space $\mathbb{H}_{\mathfrak{X}}^{\leq N} = \mathbb{H}_{\mathfrak{X}}^{(0)} \oplus \mathbb{H}_{\mathfrak{X}}^{(1)} \oplus \dots \oplus \mathbb{H}_{\mathfrak{X}}^{(N)}$ coincides with the closure in $\mathbb{H}_{\mathfrak{X}}$ of the linear span of the random variables $P_N(\mathfrak{X}(f_1), \dots, \mathfrak{X}(f_k))$, $k \geq 1$, $f_i \in \mathbf{H}$, where P_N is a polynomial of degree at most N ; cf. [14, p. 9]. Thus, the space $\mathbb{H}_{\mathfrak{X}}^{\leq N}$ does not depend on the basis in $\mathbb{H}_{\mathfrak{X}}^{(1)}$. Since $\mathbb{H}_{\mathfrak{X}}^{\leq N} = \mathbb{H}_{\mathfrak{X}}^{\leq (N-1)} \oplus \mathbb{H}_{\mathfrak{X}}^{(N)}$, the space $\mathbb{H}_{\mathfrak{X}}^{(N)}$ does not depend on the basis as well. \square

In the case of white noise \mathfrak{B} over \mathbf{H} , an orthonormal basis in $\mathbb{H}_{\mathfrak{B}}^{(1)}$ is closely related to an orthonormal basis in \mathbf{H} .

Proposition 3.4. *Let \mathfrak{B} be a white noise over a separable Hilbert space \mathbf{H} and let $\{m_1, m_2, \dots\}$ be an orthonormal basis in \mathbf{H} . Then $\{\xi_k = \mathfrak{B}(m_k), k \geq 1\}$ is an orthonormal basis in $\mathbb{H}_{\mathfrak{B}}^{(1)}$ and, for every $f \in \mathbf{H}$,*

$$(3.5) \quad \mathfrak{B}(f) = \sum_{k=1}^{\infty} (f, m_k)_{\mathbf{H}} \mathfrak{B}(m_k).$$

Proof. Note that $\mathbb{E}(\xi_k \xi_n) = \mathbb{E}(\mathfrak{B}(m_k) \mathfrak{B}(m_n)) = (m_k, m_n)_{\mathbf{H}}$, so the system $\{\xi_k, k \geq 1\}$ is orthonormal in $\mathbb{H}_{\mathfrak{B}}^{(1)}$ if and only if $\{m_k, k \geq 1\}$ is orthonormal in \mathbf{H} . If $\xi \in \mathbb{H}_{\mathfrak{B}}^{(1)}$, then $\xi = \mathfrak{B}(f)$ for some $f \in \mathbf{H}$. By assumption, $f = \sum_{k=1}^{\infty} (f, m_k)_{\mathbf{H}} m_k$, which implies (3.5) and completes the proof. \square

If $\mathbf{H} = L_2((0, T))$ and $f = \chi_t$, then (3.5) becomes a familiar representation of the standard Brownian motion on $[0, T]$:

$$(3.6) \quad W(t) = \sum_{k=1}^{\infty} \left(\int_0^t m_k(s) ds \right) \left(\int_0^T m_k(s) dW(s) \right).$$

Now, let \mathfrak{X} be a zero-mean generalized Gaussian field over a separable Hilbert space \mathbf{H} . By (3.5) and Theorem 2.4(a), we can take a white noise representation of \mathfrak{X} , $\mathfrak{X}(f) = \mathfrak{B}(\mathcal{K}^* f)$, and get an expansion of $\mathfrak{X}(f)$ using an orthonormal basis in \mathbf{H} :

$$(3.7) \quad \mathfrak{X}(f) = \sum_{k=1}^{\infty} (\mathcal{K}^* f, m_k)_{\mathbf{H}} \mathfrak{B}(m_k).$$

When $\mathbf{H} = L_2((0, T))$ and $f = \chi_t$, the associated process has representation $X(t) = \int_0^T K_{\mathfrak{X}}(t, s) dW(s)$, where $K_{\mathfrak{X}}(t, s) = (\mathcal{K}^* \chi_t)(s)$, and we get a generalization of (3.6):

$$(3.8) \quad X(t) = \sum_{k=1}^{\infty} \left(\int_0^t (\mathcal{K} m_k)(s) ds \right) \left(\int_0^T m_k(s) dW(s) \right);$$

note that $\int_0^t (\mathcal{K} m_k)(s) ds = \int_0^T K_{\mathfrak{X}}(t, s) m_k(s) ds$.

Alternatively, by Theorem 2.4(b) and Remark 2.5(b), \mathfrak{X} is a white noise over the space $\mathbf{H}_{\mathcal{R}}$ corresponding to the covariance operator \mathcal{R} of \mathfrak{X} . If $\{\overline{m}_k, k \geq 1\}$ is an orthonormal basis in $\mathbf{H}_{\mathcal{R}}$, then we have the following analog of (3.5):

$$(3.9) \quad \mathfrak{X}(f) = \sum_{k=1}^{\infty} (\mathcal{R} f, \overline{m}_k)_{\mathbf{H}} \mathfrak{X}(\overline{m}_k).$$

If \mathfrak{X} is non-degenerate, which means $\ker(\mathcal{R}) = 0$, then (3.7) and (3.9) are equivalent. Indeed, by Theorem 2.4(b), \mathbf{H} is dense in $\mathbf{H}_{\mathcal{R}}$ and we can extend \mathcal{K}^* to a bounded linear operator from $\mathbf{H}_{\mathcal{R}}$ to \mathbf{H} , because $\|\mathcal{K}^* f\|_{\mathbf{H}}^2 = \|f\|_{\mathbf{H}_{\mathcal{R}}}^2$. Clearly, (3.7) and (3.9) coincide for $f \in \mathbf{H}$, since $\{\mathcal{K}^* \overline{m}_k, k \geq 1\}$ is an orthonormal basis in \mathbf{H} . Extending (3.7) to $f \in \mathbf{H}_{\mathcal{R}}$ makes (3.7) equivalent to (3.9).

We conclude the section with a brief discussion of the Wick product, as we will need this product to define $\mathfrak{X}(f)$ for random f .

To motivate the definition of the Wick product, we make the following observation. The ordinary powers x^n have the property $x^n x^m = x^{m+n}$. By Theorem 3.2, the natural building blocks of the chaos space $\mathbb{H}_{\mathfrak{X}}$ are not the ordinary powers but Hermite polynomials of the basis elements in $\mathbb{H}_{\mathfrak{X}}^{(1)}$. It is therefore convenient to have an operation, which we denote by \diamond and call the **Wick product**, so that, for every $\xi \in \mathbb{H}_{\mathfrak{X}}^{(1)}$,

$$(3.10) \quad H_n(\xi) \diamond H_m(\xi) = H_{m+n}(\xi).$$

In fact, together with Theorem 3.2, relation (3.10) completely defines the Wick product in $\mathbb{H}_{\mathfrak{X}}$, because if $\{\xi_\alpha, \alpha \in \mathcal{I}\}$ is an orthonormal basis in $\mathbb{H}_{\mathfrak{X}}$, as defined by (3.3), then, for $\alpha = \{\alpha_k, k \geq 1\}$ and $\beta = \{\beta_k, k \geq 1\}$ we have

$$(3.11) \quad \xi_\alpha \diamond \xi_\beta = \sqrt{\frac{(\alpha + \beta)!}{\alpha! \beta!}} \xi_{\alpha + \beta},$$

where $\alpha + \beta = \{\alpha_k + \beta_k, k \geq 1\}$ and $\alpha! = \prod_{k \geq 1} \alpha_k! = \alpha_1! \alpha_2! \alpha_3! \cdots$. Using (3.11) and linearity, we now define the Wick product of two arbitrary elements of $\mathbb{H}_{\mathfrak{X}}$,

$$(3.12) \quad \left(\sum_{\alpha \in \mathcal{I}} c_\alpha \xi_\alpha \right) \diamond \left(\sum_{\beta \in \mathcal{I}} d_\beta \xi_\beta \right) = \sum_{\alpha, \beta \in \mathcal{I}} c_\alpha d_\beta \sqrt{\frac{(\alpha + \beta)!}{\alpha! \beta!}} \xi_{\alpha + \beta},$$

as long as the series on the right hand side converges in $\mathbb{H}_{\mathfrak{X}}$. In general, there is no guarantee that, for $\xi, \eta \in \mathbb{H}_{\mathfrak{X}}$, the Wick product $\xi \diamond \eta$ belongs to $\mathbb{H}_{\mathfrak{X}}$. For example, let $\xi \in \mathbb{H}_{\mathfrak{X}}^{(1)}$, $\mathbb{E}\xi^2 = 1$, and $\eta = \sum_{n=1}^{\infty} H_n(\xi)/(n\sqrt{n!})$. Then, treating ξ as the first element of the orthonormal basis in $\mathbb{H}_{\mathfrak{X}}^{(1)}$, we have $\xi = \xi_1 = \xi_{\epsilon_1}$ and $\eta = \sum_{n \geq 1} n^{-1} \xi_{n\epsilon_1}$. Then, by (3.11), $\xi_1 \diamond \xi_{n\epsilon_1} = \sqrt{n+1} \xi_{(n+1)\epsilon_1}$, so that

$$\xi \diamond \eta = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n} \xi_{(n+1)\epsilon_1},$$

and the series does not converge in $\mathbb{H}_{\mathfrak{X}}$.

Note that, unlike the usual product, the Wick product of two random variables must be computed using the chaos expansion (3.12). The lack of an easy criterion for the convergence in (3.12) is one reason for considering weighted chaos spaces. In the case when \mathfrak{X} is a white noise over $L_2(\mathbb{R}^n)$, weighted chaos spaces are described, for example, in the books [9] and [10] (see also [13]). The extension of these spaces to other Gaussian fields is straightforward, but is outside the scope of our discussion.

Let us summarize the main properties of the Wick product:

- $\xi \diamond \eta = \eta \diamond \xi$;
- $\xi \diamond (\eta \diamond \zeta) = (\xi \diamond \eta) \diamond \zeta$;
- $\xi \diamond (\eta + \zeta) = \xi \diamond \eta + \xi \diamond \zeta$;
- $\xi \diamond \eta = \xi \eta$ if $\xi, \eta \in \mathbb{H}_{\mathfrak{X}}^{(1)}$ and $\mathbb{E}(\xi \eta) = 0$.
- $\xi \diamond \eta = \xi \eta$ if either ξ or η is an element of $\mathbb{H}_{\mathfrak{X}}^{(0)}$, that is, non-random.

Similar to ordinary powers, we define Wick powers of a random variable $\eta \in \mathbb{H}_{\mathfrak{X}}$: $\eta^{\diamond n} = \eta \diamond \cdots \diamond \eta$. Replacing ordinary powers with Wick powers in a Taylor series for a function f leads to the notion of a Wick function f^{\diamond} . For example, the Wick exponential $e^{\diamond \eta}$ is defined by

$$(3.13) \quad e^{\diamond \eta} = \sum_{n=1}^{\infty} \frac{\eta^{\diamond n}}{n!}$$

and satisfies $e^{\diamond(\xi+\eta)} = e^{\diamond \xi} \diamond e^{\diamond \eta}$. If $\eta \in \mathbb{H}_{\mathfrak{X}}^{(1)}$, then direct computations show that

$$(3.14) \quad e^{\diamond \eta} = e^{\eta - \frac{1}{2}\mathbb{E}\eta^2}.$$

For more information on the Wick functions, see [10].

Just as the chaos decomposition (3.4), the Wick product does not depend on the choice of the orthonormal basis in $\mathbb{H}_{\mathfrak{X}}^{(1)}$. For example, if η_1, \dots, η_k are elements of $\mathbb{H}_{\mathfrak{X}}^{(1)}$ and m_1, \dots, m_k are non-negative integers, then $\eta_1^{\diamond m_1} \diamond \cdots \diamond \eta_k^{\diamond m_k}$ is the orthogonal projection of $\prod_{j=1}^k \eta_j^{m_j}$ onto $\mathbb{H}_{\mathfrak{X}}^{(N)}$, where $N = m_1 + \cdots + m_k$. For more details, we refer to [9, 10, 14].

4. STOCHASTIC INTEGRATION

In the definition of a generalized random field \mathfrak{X} over a Hilbert space \mathbf{H} , we consider random variables $\mathfrak{X}(f)$ for non-random $f \in \mathbf{H}$. In this section, we define $\mathfrak{X}(\eta)$ for \mathbf{H} -valued random elements η .

As a motivation, consider a white noise \mathfrak{B} over $L_2((0, T))$. By Theorem 2.6, $W(t) = \mathfrak{B}(\chi_t)$ is a standard Brownian motion; according to (3.6),

$$(4.1) \quad W(t) = \sum_{k=1}^{\infty} M_k(t) \xi_k,$$

where $\xi_k = \mathfrak{B}(m_k)$, $M_k(t) = \int_0^t m_k(s) ds$, and $\{m_k, k \geq 1\}$ is an orthonormal basis in $L_2((0, T))$. Being a continuous function, $W = W(t)$ is an element of $L_2((0, T))$. To define \mathfrak{B} on W using (4.1), one possibility is to set $\mathfrak{B}^{\circ}(W) = \sum_{k=1}^{\infty} \xi_k \mathfrak{B}(M_k)$; then direct computations show that $\sum_{k=1}^{\infty} \xi_k \mathfrak{B}(M_k) = W^2(T)/2$. In other words, $\mathfrak{B}^{\circ}(W) = \int_0^T W(t) \circ dW(t)$, where \circ denotes the Stratonovich integral. Another possibility is to set $\mathfrak{B}^{\diamond}(W) = \sum_{k=1}^{\infty} \xi_k \diamond \mathfrak{B}(M_k)$; then direct computations show that $\sum_{k=1}^{\infty} \xi_k \diamond \mathfrak{B}(M_k) = (\mathfrak{B}(T)^2 - T)/2$. In other words, $\mathfrak{B}^{\diamond}(W) = \int_0^T W(t) dW(t)$, the Itô integral.

We will now use this example to define stochastic integrals with respect to a white noise \mathfrak{B} over a separable Hilbert space \mathbf{H} . Let $\{m_k, k \geq 1\}$ be an orthonormal basis in \mathbf{H} . Define $\xi_k = \mathfrak{B}(m_k)$ and ξ_{α} , $\alpha \in \mathcal{I}$, according to (3.3).

Definition 4.5. *An \mathbf{H} -valued random element η is called $(\mathfrak{B}, \mathbf{H})$ -admissible if $\mathbb{E}\|\eta\|_{\mathbf{H}}^2 < \infty$ and, for every $f \in \mathbf{H}$, the random variable $(\eta, f)_{\mathbf{H}}$ is $\mathcal{F}^{\mathfrak{B}}$ -measurable.*

By Theorem 3.2 and Proposition 3.4, every $(\mathfrak{B}, \mathbf{H})$ -admissible η has chaos expansion

$$(4.2) \quad \eta = \sum_{\alpha \in \mathcal{I}} \eta_\alpha \xi_\alpha, \quad \eta_\alpha = \mathbb{E}(\eta \xi_\alpha) \in \mathbf{H}.$$

Definition 4.6. Let η be $(\mathfrak{B}, \mathbf{H})$ -admissible with chaos expansion (4.2).

The Itô stochastic integral of η with respect to \mathfrak{B} is

$$(4.3) \quad \mathfrak{B}^\diamond(\eta) = \sum_{\alpha \in \mathcal{I}} \mathfrak{B}(\eta_\alpha) \diamond \xi_\alpha,$$

where \diamond is the Wick product. The Stratonovich stochastic integral of η with respect to \mathfrak{B} is

$$(4.4) \quad \mathfrak{B}^\circ(\eta) = \sum_{\alpha \in \mathcal{I}} \mathfrak{B}(\eta_\alpha) \cdot \xi_\alpha,$$

where \cdot is the usual product.

Since every generalized Gaussian field and every Gaussian process can be represented as a white noise over a suitable Hilbert space, formulas (4.3) and (4.4) define stochastic integral with respect to any Gaussian process or field. We will see below that these formulas also provide a chaos expansion of the integral in terms of the chaos expansion of the integrand; note that neither (4.3) nor (4.4) is a chaos expansion in the sense of (4.2). The two immediate questions that are raised by the above definition and will be discussed below are (a) the convergence of the series, and (b) the dependence of the integrals on the choice of the basis in \mathbf{H} .

We start by deriving the chaos expansion of the integrals without investigating the question of convergence.

Theorem 4.7. Let η be $(\mathfrak{B}, \mathbf{H})$ -admissible with chaos expansion (4.2), and assume that

$$(4.5) \quad \eta_\alpha = \sum_{k=1}^{\infty} \eta_{\alpha,k} m_k.$$

Then

$$(4.6) \quad \mathfrak{B}^\diamond(\eta) = \sum_{\alpha \in \mathcal{I}} \left(\sum_{k=1}^{\infty} \sqrt{\alpha_k} \eta_{\alpha-\epsilon_k, k} \right) \xi_\alpha,$$

$$(4.7) \quad \mathfrak{B}^\circ(\eta) = \sum_{\alpha \in \mathcal{I}} \left(\sum_{k=1}^{\infty} (\sqrt{\alpha_k} \eta_{\alpha-\epsilon_k, k} + \sqrt{\alpha_k + 1} \eta_{\alpha+\epsilon_k, k}) \right) \xi_\alpha.$$

Proof. By (4.5) and linearity,

$$\mathfrak{B}(\eta_\alpha) = \sum_{k=1}^{\infty} \eta_{\alpha,k} \mathfrak{B}(m_k) = \sum_{k=1}^{\infty} \eta_{\alpha,k} \xi_k.$$

Therefore,

$$(4.8) \quad \mathfrak{B}^\diamond(\eta) = \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \eta_{\alpha,k} \xi_k \diamond \xi_\alpha = \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \sqrt{\alpha_k + 1} \eta_{\alpha,k} \xi_{\alpha+\epsilon_k},$$

where the last equality follows from (3.11); recall that ϵ_k is the multi-index with the only non-zero entry, equal to one, at position k . By shifting the summation index, we get (4.6). Note that, for every $\alpha \in \mathcal{I}$, the inner sum in (4.6) contains finitely many non-zero terms.

To establish (4.7), we write, similar to (4.8),

$$\mathfrak{B}^\circ(\eta) = \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \eta_{\alpha,k} \xi_k \xi_\alpha,$$

and, instead of (3.11), use the following property of the Hermite polynomials,

$$H_1(x)H_n(x) = H_{n+1}(x) + nH_{n-1}(x),$$

which implies

$$(4.9) \quad \xi_k \xi_\alpha = \left(\prod_{j \neq k} \frac{H_{\alpha_j}(\xi_j)}{\sqrt{\alpha_j!}} \right) \frac{H_1(\xi_k) H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} = \sqrt{\alpha_k + 1} \xi_{\alpha + \epsilon_k} + \sqrt{\alpha_k} \xi_{\alpha - \epsilon_k},$$

and then (4.7) follows. \square

Now, let us address the questions of convergence and independence of basis. The Cauchy-Schwartz inequality implies that if

$$(4.10) \quad \sum_{\alpha \in \mathcal{I}} |\alpha| \|\eta_\alpha\|_{\mathbf{H}}^2 < \infty,$$

then $\mathfrak{B}^\circ(\eta) \in \mathbb{H}_{\mathfrak{B}}$. Further examination of (4.6) shows that, for every $(\mathfrak{B}, \mathbf{H})$ -admissible η satisfying (4.10), $\mathfrak{B}^\circ(\eta)$ coincides with the action of the divergence operator (adjoint of the Malliavin derivative, see [15]) on η and therefore does not depend on any arbitrary choices, such as the basis in \mathbf{H} . In particular, if $\mathbf{H} = L_2(I)$, then $\mathfrak{B}^\circ(\eta)$ is the Itô-Skorokhod integral of η . On the other hand, (4.6) allows the extension of \mathfrak{B}° to weighted chaos spaces, similar to those considered in [9, 10, 13].

For the Stratonovich integral $\mathfrak{B}^\circ(\eta)$, note that the Malliavin derivative \mathbb{D} of ξ_α satisfies

$$\mathbb{D}\xi_\alpha = \sum_{k=1}^{\infty} \sqrt{\alpha_k} \xi_{\alpha - \epsilon_k} m_k;$$

this follows directly from the definition of \mathbb{D} [15, Definition 1.2.1] and the relation $H'_n(x) = nH_{n-1}(x)$. As a result, we use (4.9) to re-write (4.7) as

$$(4.11) \quad \mathfrak{B}^\circ(\eta) = \mathfrak{B}^\circ(\eta) + \sum_{\alpha \in \mathcal{I}} (\eta_\alpha, \mathbb{D}\xi_\alpha)_{\mathbf{H}}.$$

In particular, if \mathfrak{B} is a white noise over $L_2(I)$ and $\eta = \eta(t)$ is in the domain of the Malliavin derivative, then

$$(4.12) \quad \mathfrak{B}^\circ(\eta) = \mathfrak{B}^\circ(\eta) + \int_I \mathbb{D}_t \eta dt,$$

where

$$\mathbb{D}_t \eta = \sum_{\alpha \in \mathcal{I}} \eta_\alpha(t) \left(\sum_{k=1}^{\infty} \sqrt{\alpha_k} \xi_{\alpha - \epsilon_k} m_k(t) \right);$$

Unlike the Itô integral, though, condition (4.10) is not enough to ensure the existence of $\mathfrak{B}^\circ(\eta)$ as an element of $\mathbb{H}_{\mathfrak{X}}$. When \mathbf{H} is the Hilbert space of functions on an interval I , square integrable with respect to a (not necessarily Lebesgue) measure μ , the sufficient conditions for the Stratonovich integrability are discussed in [15, Chapter 3]. Alternatively, \mathfrak{B}° can be defined in weighted chaos spaces, but the details of the construction have yet to be worked out.

In what follows, we will concentrate on the Itô integral.

Let \mathfrak{X} be a zero-mean non-degenerate generalized Gaussian field over a separable Hilbert space \mathbf{H} . As we mentioned earlier, by the second part of Theorem 2.4, \mathfrak{X} is a white noise over a bigger Hilbert space $\mathbf{H}_{\mathcal{R}}$, and then $\mathfrak{X}^\circ(\eta)$ can be defined using (4.3). If the space $\mathbf{H}_{\mathcal{R}}$ is difficult to describe, one can use representation (2.5) from the first part of Theorem 2.4 and consider a different formula for the stochastic integral:

$$(4.13) \quad \mathfrak{X}^\circ(\eta) = \mathfrak{B}^\circ(\mathcal{K}^*\eta)$$

for every $(\mathfrak{B}, \mathbf{H})$ -admissible η . Similar to the non-random integrands, the two definitions are equivalent if \mathfrak{X} is non-degenerate.

Unlike (4.3), representation (4.13) is not intrinsic: the operator \mathcal{K}^* and the white noise \mathfrak{B} are not uniquely determined by \mathfrak{X} . On the other hand, in many examples, such as fractional Brownian motion with the Hurst parameter bigger than $1/2$, it is possible to take $\mathbf{H} = L_2(I)$, and then (4.13) becomes more convenient than (4.3). To derive the chaos expansion of $\mathfrak{X}^\circ(\eta)$ using (4.13), fix an orthonormal basis $\{m_k, k \geq 1\}$ in \mathbf{H} , define $\xi_k = \mathfrak{B}(m_k)$, and consider the corresponding orthonormal basis $\{\xi_\alpha, \alpha \in \mathcal{I}\}$ in $\mathbb{H}_{\mathfrak{B}}$ constructed according to (3.3). It follows from (4.6) that

$$(4.14) \quad \mathfrak{X}^\circ(\eta) = \sum_{\alpha \in \mathcal{I}} \left(\sum_{k=1}^{\infty} \sqrt{\alpha_k} \tilde{\eta}_{\alpha - \epsilon_k, k} \right) \xi_\alpha,$$

where

$$\tilde{\eta}_{k, \alpha} = \mathbb{E}((\mathcal{K}^*\eta, m_k)_{\mathbf{H}} \xi_\alpha).$$

If $H = L_2(I)$, then (4.14) becomes

$$(4.15) \quad \mathfrak{X}_t^\circ(\eta) = \sum_{\alpha \in \mathcal{I}} \left(\sum_{k \geq 1} \sqrt{\alpha_k} \left(\int_I \eta_{\alpha - \epsilon_k}(t) (\mathcal{K}m_k)(t) dt \right) \right) \xi_\alpha,$$

where $\eta_\alpha(t) = \mathbb{E}(\eta(t) \xi_\alpha)$. In this case, by analogy with the Brownian motion, $\int_0^t \eta(s) dX(s)$ can be an alternative notation for $\mathfrak{X}_t^\circ(\eta)$, where $X(t)$ is the associated process of \mathfrak{X} .

We conclude this section with a brief discussion of stochastic differential equations. To introduce the time evolution, we use the function χ_t , the characteristic function of the interval $[0, t]$, and define time-dependent stochastic integrals

$$(4.16) \quad \mathfrak{B}_t^\circ(\eta) := \mathfrak{B}^\circ(\eta \chi_t), \quad \mathfrak{X}_t^\circ(\eta) := \mathfrak{X}^\circ(\eta \chi_t).$$

These definitions put an obvious restriction on the Hilbert space \mathbf{H} , which we call Property I: \mathbf{H} is a collection of function or generalized functions and, for every $\eta \in \mathbf{H}$

and every fixed t , the (point-wise) product $\eta\chi_t$ is defined and belongs to \mathbf{H} . There is a more significant restriction on \mathbf{H} , which we illustrate on the following equation:

$$(4.17) \quad u(t) = 1 + \mathfrak{B}_t^\diamond(u), \quad 0 \leq t \leq T,$$

where \mathfrak{B} is white noise over a Hilbert space \mathbf{H} with Property I. Let us assume that the solution belongs to $\mathbb{H}_{\mathfrak{B}}$ so that $u(t) = \sum_{\alpha \in \mathcal{I}} u_\alpha(t) \xi_\alpha$ and each u_α is an element of \mathbf{H} . By (4.16), we can re-write (4.17) as

$$(4.18) \quad u(t) = 1 + \mathfrak{B}^\diamond(u\chi_t),$$

and then (4.6) implies

$$(4.19) \quad u_\alpha(t) = 1 + \sum_{k=1}^{\infty} (u_{\alpha-\epsilon_k} \chi_t, m_k)_{\mathbf{H}}.$$

Thus, the expression $(u_{\alpha-\epsilon_k} \chi_t, m_k)_{\mathbf{H}}$, as a function of t , must be an element of \mathbf{H} , and the Hilbert space \mathbf{H} must have another special property, which we call Property II: *for every $f, g \in \mathbf{H}$, the inner product $(f\chi_t, g)_{\mathbf{H}}$, as a function of t , is an element of \mathbf{H}* . By the Cauchy-Schwartz inequality, the space $L_2(I, \mu)$, with $\mu(I) < \infty$, has both Property I and Property II. Representation (4.13) then allows us to analyze stochastic equations for certain generalized Gaussian fields. This analysis should be a subject of a separate paper, and below we consider only one particular example.

Theorem 4.8. *If \mathfrak{X} is a zero-mean generalized Gaussian field over $L_2((0, T))$, then the solution of the equation*

$$(4.20) \quad u(t) = 1 + \mathfrak{X}_t^\diamond(u)$$

is unique in $L_2((0, T); \mathbb{H}_{\mathfrak{X}})$ and is given by

$$(4.21) \quad u(t) = e^{\diamond X(t)},$$

where e^\diamond is the Wick exponential function (3.13) and $X(t) = \mathfrak{X}(\chi_t)$ is the associated process of \mathfrak{X} .

Proof. Let $\mathfrak{X}(f) = \mathfrak{B}(\mathcal{K}^* f)$ be a white noise representation of \mathfrak{X} over $L_2((0, T))$. We start by establishing uniqueness of solution in $L_2((0, T); \mathbb{H}_{\mathfrak{B}})$, which, because of the inclusion $\mathbb{H}_{\mathfrak{X}} \subseteq \mathbb{H}_{\mathfrak{B}}$, is even stronger. By linearity, the difference $Y(t)$ of two solutions of (4.20) satisfies $Y(t) = \mathfrak{X}_t^\diamond(Y)$. If $Y(t) = \sum_{\alpha \in \mathcal{I}} y_\alpha(t) \xi_\alpha$, then (4.15) implies

$$(4.22) \quad y_\alpha(t) = \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t y_{\alpha-\epsilon_k}(s) \tilde{m}_k(s) ds,$$

where $\tilde{m}_k = \mathcal{K}m_k$. In particular, if $|\alpha| = 0$, then $y_\alpha(t) = 0$ for all t . By induction on $|\alpha|$, $y_\alpha(t) = 0$ for all $\alpha \in \mathcal{I}$: if $y_\alpha = 0$ for all α with $|\alpha| = n$, then, since $|\alpha - \epsilon_k| = |\alpha| - 1$, equality (4.22) implies $y_\alpha = 0$ for all α with $|\alpha| = n + 1$.

To establish (4.21), let

$$\widetilde{M}_k(t) = \int_0^t (\mathcal{K}m_k)(s) ds.$$

By (3.8),

$$X(t) = \sum_{k=1}^{\infty} \widetilde{M}_k(t) \xi_k,$$

and, because of the independence of ξ_k for different k ,

$$e^{\diamond X(t)} = \prod_{k \geq 1} e^{\diamond \widetilde{M}_k(t) \xi_k} = \sum_{\alpha \in \mathcal{I}} \frac{\widetilde{M}^\alpha(t)}{\sqrt{\alpha!}} \xi_\alpha,$$

where

$$\widetilde{M}^\alpha(t) = \prod_{k=1}^{\infty} \widetilde{M}_k^{\alpha_k}(t).$$

Similar to (4.22), we conclude that if the solution $u = u(t)$ has the chaos expansion $u(t) = \sum_{\alpha \in \mathcal{I}} u_\alpha(t) \xi_\alpha$, then $u_\alpha(t) = 1$ if $|\alpha| = 0$ and

$$(4.23) \quad u_\alpha(t) = \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t u_{\alpha - \epsilon_k}(s) \widetilde{m}_k(s) ds,$$

if $|\alpha| > 0$. Then direct computations show that

$$u_\alpha(t) = \frac{\widetilde{M}^\alpha(t)}{\sqrt{\alpha!}}, \quad |\alpha| \geq 1,$$

satisfies (4.23):

$$\begin{aligned} \frac{du_\alpha(t)}{dt} &= \frac{1}{\sqrt{\alpha!}} \frac{d}{dt} \prod_{k=1}^{\infty} \widetilde{M}_k^{\alpha_k}(t) = \frac{1}{\sqrt{\alpha!}} \sum_{k=1}^{\infty} \alpha_k \widetilde{M}_k^{\alpha_k-1}(t) \widetilde{m}_k(t) \prod_{j \neq k} \widetilde{M}_j^{\alpha_j}(t) \\ &= \sum_{k=1}^{\infty} \sqrt{\alpha_k} \widetilde{m}_k(t) \frac{\widetilde{M}^{\alpha - \epsilon_k}(t)}{\sqrt{(\alpha - \epsilon_k)!}} = \sum_{k=1}^{\infty} \sqrt{\alpha_k} m_k(t) u_{\alpha - \epsilon_k}(t). \end{aligned}$$

□

Theorem 4.8 is a generalization of the familiar result that the geometric Brownian motion $u(t) = e^{W(t) - (t/2)} = e^{\diamond W(t)}$ satisfies $u(t) = 1 + \int_0^t u(s) dW(s)$: by (4.21) and (3.14), for a class of zero-mean Gaussian processes $X = X(t)$ with covariance function $R(t, s)$, and with a suitable interpretation of the stochastic integral, the solution of the equation $u(t) = 1 + \int_0^t u(s) dX(s)$ is

$$u(t) = e^{X(t) - \frac{1}{2} R(t, t)}.$$

The proof of the theorem suggests that stochastic equations in the Itô-Skorokhod sense are more suitable for analysis using chaos expansion than the equations in the Stratonovich sense. Indeed, equation (4.20) leads to the system of equations (4.23) that is solvable by induction on $|\alpha|$. By contrast, equation $u(t) = 1 + \mathfrak{X}_t^\circ(u)$ leads to a system that is not solvable by induction on $|\alpha|$: according to (4.7), u_α will depend on both $u_{\alpha - \epsilon_k}$ and $u_{\alpha + \epsilon_k}$.

The arguments used in the proof of Theorem 4.8 can be extended to more general linear equations and to generalized fields over $L_2((0, T), \mu)$ for different measures μ , although the precise results will essentially depend on certain fine properties of μ .

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